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APPLICATION OF HOMOGENIZATION METHOD TO JUSTIFICATION OF 1-D MODEL FOR BEAM OF PERIODIC STRUCTURE HAVING INITIAL STRESSES

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Abstract—In this paper a 1-D model of the beam theory is derived from the 3-D elasticity theory problem for a beam having initial stresses. We consider a thin inhomogeneous beam of periodic structure whose diameter and period are comparable in order (the classical approach cannot be applied). The transition from the 3-D elasticity theory problem to the 1-D beam problem is made on the basis of a modified two-scale expansion homogenization method without any simplified assumptions. The obtained 1-D modes correspond to the model of the beam subjected to the axial force and beam subjected to moments. The obtained models are in agreement with the classical models for homogeneous cylindrical beams. © 1998 Elsevier Science Ltd. All rights reserved.

INTRODUCTION

The asymptotic homogenization method, widely used for monolithic composites (see e.g. Bensoussan *et al.*, 1978; Sanchez-Palencia, 1980; Bakhvalov and Panasenko, 1989; Aboudi, 1991; Nemat-Naser and Hori, 1993; Kalamkarov and Kolpakov, 1997; and references in these books) are now being used to study non-homogeneous bodies occupying thin regions—plates and beams. In the works by Ciarlet and Destuyner (1979), Kohn and Vogelius (1984), Caillerie (1984), Panasenko and Reztsov (1987), transition from a 3-D elasticity problem to plate theory problem was made, in the paper by Kolpakov (1991), transition from a 3-D elasticity problem to beam theory was made.

In all the papers above the 3-D problem with no initial stresses were studied. In the present paper transition from a 3-D elasticity problem with initial stresses to a beam theory problem is made. It is made on the basis of a two-scale asymptotic method used earlier for monolithic composites and plates (see the references above) and modified for beam in the above mentioned paper by Kolpakov (1991).

The accounting of the initial stresses plays an important role in the mechanics of structures, in particular, in the stability of structures. Then, the deriving of an asymptotically exact model of a non-homogeneous beam with initial stresses seems to be practically important.

We will consider the non-homogeneous beam of periodic structure (which are widely used in modern structures). The non-homogeneity of the beam can be a result of both the non-homogeneity of the material the beam is made of, and non-homogeneity of the beam geometry. Both cases will be covered by our consideration. The classical uniform beam is the partial case of the beam under consideration.

Earlier the homogenization problem for bodies with initial stresses was studied for monolithic composites in the works of Kolpakov (1990, 1992). The homogenization problem for plate with initial stresses was studied in the work of Kolpakov (1987) on the basis of the two-dimensional plate equations (i.e. for the plate having a thickness much smaller than the dimension of inhomogeneities).

As it will be seen, the analysis in the case under consideration differs both from the analysis for monolithic composites with initial stresses and thin bodies with no initial stresses. The difference between the given problem and the problems examined in the works considering the structures with no initial stresses (see Bensoussan *et al.* 1987; Sanchez-Palencia, 1980; Kohn and Vogelius, 1984; Caillerie, 1984; Kolpakov, 1991; Kalamkarov and Kolpakov, 1997) is the asymmetry of the coefficients, requiring more detailed analysis of the problem. The difference between the given problem and the problems for a stressed monolith body examined in the works by Kolpakov (1990, 1992), is connected with the difference of asymptotic expansions for the monolithic body, plate and beam (see Bensoussan *et al.*, 1978; Caillerie, 1984; Kolpakov, 1991; Kalamkarov and Kolpakov, 1997).

As it will be seen from the discussion below, the order of the initial stresses relative to the characteristic diameter of the beam ε, plays a significant role in the problem. To account for this, we take the initial stresses σ_{ij}^* in the form $\sigma_{ij}^* = \varepsilon^{-3}\sigma_{ij}^{*(-3)} + \varepsilon^{-2}\sigma_{ij}^{*(-2)}$.

1. FORMULATION OF THE PROBLEM

We will examine a body of periodic structure obtained by repeating a certain small periodicity cell (PC) P_ε among the $0x_1$-axis (Fig. 1). Here ε is the PC characteristic dimension, which is assumed to be small (that is formalized in the form $\varepsilon \to 0$). As a result, we have a body of periodic structure with the small diameter—a 3-D beam. For $\varepsilon \to 0$ the 3-D beam "tightens" to the segment $[-a, a]$ at the $0x_1$-axis (Fig. 1)—a 1-D beam. Our aim is to derive a model describing the 1-D beam.

The starting point of our investigation is the exact 3-D formulation of an elasticity problem for a body with initial stresses without any simplified assumptions, presented in the book by Washizu (1982). In accordance with this model, the equilibrium equations of the beam as a 3-D elastic body with initial stresses can be written in the form

$$\partial/\partial x_j(A_{ijmn}(x_1, \mathbf{x}/\varepsilon)\, \partial u_m^\varepsilon/\partial x_n) = 0 \quad \text{in } Q_\varepsilon. \tag{1.1}$$

The boundary conditions can be written in the form

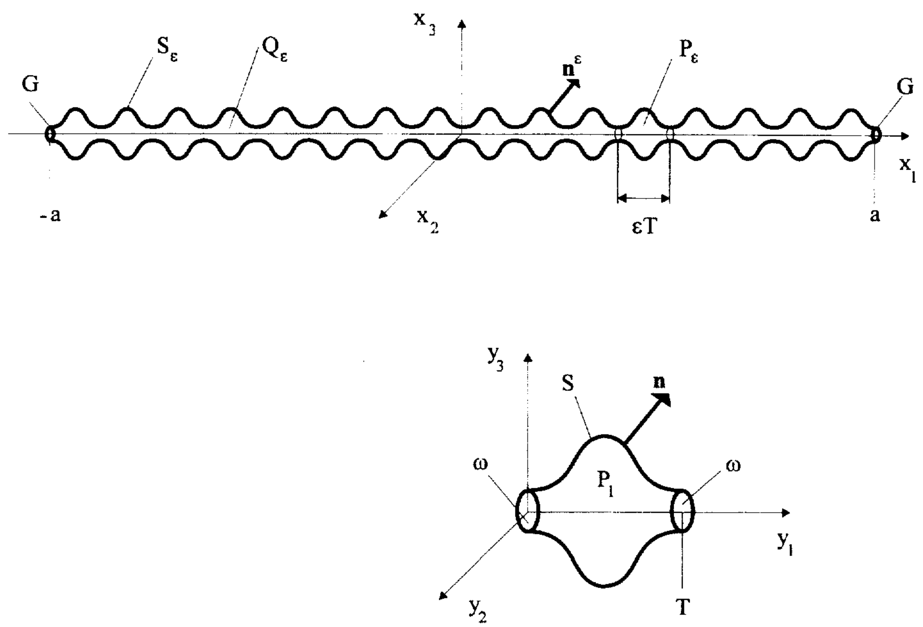

Fig. 1. The beam of periodic structure and those periodicity cells in the "fast" variables $y = x/\varepsilon$.

$$A_{ijmn}(x_1, \mathbf{x}/\varepsilon)\,\partial u^{\varepsilon}_m/\partial x_n n^{\varepsilon}_j = 0 \quad \text{on } S_\varepsilon, u^{\varepsilon}(\mathbf{x}) = 0 \quad \text{on } G. \tag{1.2}$$

Here Q_ε is the region occupied by the beam; S_ε is the free surface of the body, $\mathbf{n}^\varepsilon$ is a normal to S_ε; the body is fastened to the surface G (see Fig. 1); $\mathbf{u}^\varepsilon$ are the displacements; A_{ijmn} are known (see Washizu, 1982) combinations of the tensor of elasticity constants $\varepsilon^{-4}a_{ijmn}$ and the initial stresses σ^*_{ij}, which, in the case under consideration, are taken in the form

$$A_{ijmn}(x_1, \mathbf{x}/\varepsilon) = \varepsilon^{-4}a_{ijmn}(\mathbf{x}/\varepsilon) + \varepsilon^{-3}b^{(-3)}_{ijmn}(x_1, \mathbf{x}/\varepsilon) + \varepsilon^{-2}b^{(-2)}_{ijmn}(x_1, \mathbf{x}/\varepsilon), \tag{1.3}$$

where

$$b^{(p)}_{ijmn}(x_1, \mathbf{x}/\varepsilon) = \sigma^{*(p)}_{jn}(x_1, \mathbf{x}/\varepsilon)\,\delta_{im}, \quad p = -3, -2, \delta_{ii} = 1 \quad \text{and} \quad \delta_{im} = 0 \quad \text{if } i \neq m. \tag{1.4}$$

With the use of two-scale method (see e.g. Bensoussan *et al.*, 1978) a function $f(x_1, \mathbf{x}/\varepsilon)$ of the arguments x_1 and $\mathbf{x}/\varepsilon$ is considered as function $f(x_1, \mathbf{y})$ of "slow" variable x_1 and "fast" variables $\mathbf{y} = \mathbf{x}/\varepsilon$. In accordance with this note the functions a_{ijmn}, $\sigma^{*(p)}_{jn}$ will be written in the form $a_{ijmn}(\mathbf{y})$, $\sigma^{*(p)}_{jn}(x_1, \mathbf{y})$. The functions $a_{ijmn}(\mathbf{y})$, $\sigma^{*(p)}_{jn}(x_1, \mathbf{y})$ are periodic in y_1 with period T corresponding to the period of the beam structures (T is projection of the PC P_1 on the Oy_1-axis in the "fast" variables, see Fig. 1).

The stresses σ_{ij} determined by the formula $\sigma_{ij} = A_{ijmn}(x_1, \mathbf{y})\,\partial u^{\varepsilon}_m/\partial x_n$ are called the additional stresses (see Washizu, 1982). For $A_{ijmn}(x_1, \mathbf{y})$ determined by eqn (1.3), the relationship between additional stresses σ_{ij} and displacements u^ε takes the form:

$$\sigma_{ij} = (\varepsilon^{-4}a_{ijmn}(\mathbf{y}) + \varepsilon^{-3}b^{(-3)}_{ijmn}(x_1, \mathbf{y}) + \varepsilon^{-2}b^{(-2)}_{ijmn}(x_1, \mathbf{y}))\,\partial u^{\varepsilon}_m/\partial x_n. \tag{1.5}$$

The formula (1.5) can be considered as a local governing equation.

Note. In connection with the fact that the coefficients eqn (1.3) are written in a form different from that normally used, we will comment briefly on the terms in eqn (1.3). The tensor $\varepsilon^{-4}a_{ijmn}$ describes the elastic constants of the material the beam is made of. The multiplier ε^{-4} guarantees that the bending stiffnesses of the beam will be non-zero, as $\varepsilon \to 0$. The other terms describe the initial stresses. The term $\varepsilon^{-2}b^{(-2)}_{ijmn} = \varepsilon^{-2}\sigma^{*(-2)}_{jn}\delta_{im}$ corresponds to the tension of the beam by a force which is independent of ε. In fact, the axial force is equal to the stress multiplied by the cross-section area, which has the characteristic value equal to ε^2. In the classical theory, the beam is buckling under the tension axial force (corresponding to stresses $\varepsilon^{-2}\sigma^{*(-2)}_{11}$). If the tension force is zero, the local initial stresses in the 3-D beam can be non-zero and can cause buckling. To take into consideration this case, we introduce the term $\varepsilon^{-3}b^{(-3)}_{ijmn} = \varepsilon^{-3}\sigma^{*(-3)}_{jn}\delta_{im}$. The condition that the tension force corresponding to $\sigma^{*(-3)}_{jn}$ is zero, is written in the form:

$$\langle \sigma^{*(-3)}_{11} \rangle = 0. \tag{1.6}$$

Here $\langle * \rangle = T^{-1}\int_{P_1} *\,d\mathbf{y}$ is the average value over the PC $P_1 = \{\mathbf{y} = \mathbf{x}/\varepsilon : \mathbf{x} \in P_\varepsilon\}$ in the "fast" variables $\mathbf{y}$.

2. ASYMPTOTIC EXPANSION

We will study global deformations of the beam, in particular, buckling in global forms, as $\varepsilon \to 0$. To do this, we use the following asymptotic expansion proposed in the work by Kolpakov (1991):

Asymptotic expansion for displacements

$$\mathbf{u}^{\varepsilon} = \mathbf{u}^{(0)}(\mathbf{x}_1) + \varepsilon \mathbf{u}^{(1)}(\mathbf{x}_1, \mathbf{y}) + \cdots = \mathbf{u}^{(0)}(\mathbf{x}_1) + \sum_{k=1}^{\infty} \varepsilon^k \mathbf{u}^{(k)}(\mathbf{x}_1, \mathbf{y}), \tag{2.1}$$

asymptotic expansion for stresses

$$\sigma_{ij} = \sum_{p=-4}^{\infty} \varepsilon^p \sigma_{ij}^{(p)}(x_1, \mathbf{y}). \tag{2.2}$$

Here x_1 is a "slow" variable along the axis of the beam $[-a, a]$, $\mathbf{y} = \mathbf{x}/\varepsilon$ are the "fast" variables. The functions in the right-side of eqns (2.1) and (2.2) are assumed to be periodic in y_1 with period T. Note, that the term $u^{(0)}(x_1)$ in eqn (2.1) depends on the "slow" variable x_1 only. The expansion (2.4) is starting with a term of order of ε^{-4} in accordance with the governing eqn (1.5) and expansion (2.1) for displacements.

Analysis of the problem eqns (1.1)–(1.4) breaks down into two stages. The first entails obtaining the equations of equilibrium for the beam considered as a 1-D structure. As in the case of plates (see e.g. Caillerie, 1984), this stage is not involved with local governing equations [in the case under consideration, with eqn (1.5)], and it is the same stage for any governing equations.

There are the following equations for the forces introduced by the formula $N_{ij}^{(p)} = \langle \sigma_{ij}^{(p)} \rangle$ and the moments introduced by the formula $M_{j\beta}^{(p)} = \langle \delta_{j1}^{(p)} y_\beta \rangle$ $(i, j = 1, 2, 3$ and $\beta = 2, 3)$ derived in the work by Kolpakov (1991):

$$N_{11,1x}^{(-3)} = 0, \tag{2.3}$$

$$N_{\beta 1,1x}^{(-2)} = 0, \tag{2.4}$$

$$M_{1\beta,1x}^{(-3)} - N_{1\beta}^{(-2)} = 0, \tag{2.5}$$

$$M_{,1x} + (N_{32}^{(-2)} - N_{23}^{(-2)}) = 0. \tag{2.6}$$

Here $M_{1\beta}$ are the bending moments, $M = M_{32}^{(-3)} - M_{23}^{(-3)}$ means the turning moment. The eqns (2.3)–(2.6) coincide with the classical ones.

Here and below the Latin indices take the values 1, 2, 3 and the Greek indices take the values 2, 3; $,1x$ means $\partial/\partial x_1$ and $,jy$ means $\partial/\partial y_j$.

The following equations are satisfied for the $\sigma_{ij}^{(p)}$ from the local stress expansions (see Kolpakov, 1991):

$$\sigma_{ij,jy}^{(p)} + \sigma_{i1,1x}^{(p-1)} = 0 \quad \text{in } P_1, \quad \sigma_{ij}^{(p)} n_j = 0 \quad \text{on } S \quad p = -4, -3 \quad (\sigma_{ij}^{(-5)} = 0). \tag{2.7}$$

Here S means the lateral (free) surface of PC P_1 (P_1 was determined above), $\mathbf{n}$ is the normal to S (see Fig. 1).

One can derive the eqns (2.7) substituting the expansion (2.2) into the local equilibrium eqns (1.1), rewritten in the form $\partial \sigma_{ij}^{(p)}/\partial x_j = 0$, and then equating the terms with identical powers of ε.

Relations (2.3)–(2.7) are independent of the local governing relations.

The second stage of analysis of the problem consists of obtaining the governing equations for the beam as 1-D structure and the excluding of unknown quantities from the equilibrium equations. In contrast to the first stage, this stage does involve local governing equations and it is the main stage in our investigation.

3. CASE OF NON-ZERO AXIAL FORCE

We begin our analysis with the case $\{\sigma^{*(-3)}_{jn} = 0, \langle \sigma^{*(-2)}_{11} \rangle \neq 0\}$ corresponding to non-zero axial force.

With the use of two-scale expansion, the differential operators are presented in the form of sum of operators in $\{x_i\}$ and in $\{y_i\}$ (see Bensoussan *et al.*, 1978). For the function $f(x_1, \mathbf{y})$ of the arguments x_1 and $\mathbf{y} = (y_1, y_2, y_3)$, as in the right-hand sides of eqns (2.1) and (2.2), this representation takes the form

$$\partial f/\partial x_\alpha = \varepsilon^{-1} f_{,\alpha y} \quad (\alpha = 2, 3), \quad \partial f/\partial x_1 = f_{,1x} + \varepsilon^{-1} f_{,1y}. \tag{3.1}$$

Substituting eqns (2.1) and (2.2) into the local governing eqn (1.5), with allowance for eqn (3.1) we obtain

$$\sum_{p=-4}^{\infty} \varepsilon^p \sigma_{ij}^{(p)} = \sum_{k=0}^{\infty} \varepsilon^k (\varepsilon^{-4} a_{ijmn} + \varepsilon^{-2} b_{ijmn}^{(-2)})(u_{m,1x}^{(k)} \delta_{1n} + \varepsilon^{-1} u_{m,ny}^{(k)})$$

$$\delta_{ii} = 1 \quad \text{and} \quad \delta_{im} = 0 \quad \text{if } i \neq m. \tag{3.2}$$

Equating the terms with identical power of ε in eqn (3.2), we obtain

$$\sigma_{ij}^{(p)} = a_{ijm1} u_{m,1x}^{(p+4)} + a_{ijmn} u_{m,ny}^{(p+5)}, \quad p = -4, -3, \tag{3.3}$$

$$\sigma_{ij}^{(-2)} = a_{ijm1} u_{m,1x}^{(2)} + b_{ijm1}^{(-2)} u_{m,1x}^{(0)} + a_{ijmn} u_{m,ny}^{(3)} + b_{ijmn}^{(-2)} u_{m,ny}^{(1)}. \tag{3.4}$$

As it was assumed above

$$\mathbf{u}^{(k)}(\mathbf{y}) \text{ is periodic in } y_1 \text{ with period } T, \quad k = 1, 2, \ldots. \tag{3.5}$$

Let us consider the problems (2.7) ($p = -4$), (3.3) ($p = -4$), (3.5) ($k = 1$). Allowing for the fact that the function of the argument x_1 plays the role of a parameter in the problems in the variables $\mathbf{y}$ and $\mathbf{u}^{(0)}$ depends on x_1, only, solution of the problems (2.7) ($p = -4$), (3.3) ($p = -4$), (3.5) ($k = 1$) can be found in the form given in the work by Kolpakov (1991)

$$\mathbf{u}^{(1)} = -y_\alpha u_{\alpha,1x}^{(0)}(x_1)\mathbf{e}_1 + \mathbf{U}(y)\phi(x_1) + \mathbf{V}(x_1). \tag{3.6}$$

Here $\{\mathbf{e}_i\}$ are basis vectors of the coordinate system; $\mathbf{U}(y) = y_\Gamma s_\gamma \mathbf{e}_\gamma$ (summation in γ, Γ is assumed) where $s_1 = 0$, $s_2 = -1$, $s_3 = 1$; $\Gamma = 2$ if $\gamma = 3$ and $\Gamma = 3$ if $\gamma = 2$; $\mathbf{V}(x_1)$ is an arbitrary function of the argument x_1 (it will be determined below).

One can verify formula (3.6) by substituting eqns (3.6) into eqns (2.7) ($p = -4$), (3.3), (3.5) ($k = 1$) with allowance for eqn (3.1).

Substitution of eqn (3.6) into eqn (3.3) gives the following equations

$$\sigma_{ij}^{(-4)} = 0,$$

$$\sigma_{ij}^{(-3)} = a_{ijmn}(\mathbf{y}) u_{m,ny}^{(2)} + a_{ij11}(\mathbf{y}) y_\alpha u_{\alpha,1x1x}^{(0)}(x_1) + a_{ij11}(\mathbf{y}) V_{1,1x}(x_1) + a_{ij\gamma 1}(\mathbf{y}) s_\gamma y_\Gamma \phi_{,1x}(x_1). \tag{3.7}$$

Let us examine the problems (2.7) ($p = -3$), (3.7), (3.5) ($k = 2$). In order to solve problems of this kind, the so-called cellular problems are introduced (see e.g. Bensoussan *et al.*, 1978; Caillerie, 1984). In the case under consideration, we introduced the functions $X^{11}(y)$, $X^{2\alpha}(y)$ ($\alpha = 2, 3$) and $X^3(y)$ which are determined by solutions of the cellular problems of the beam theory (see Kolpakov, 1991):

$$(a_{ijmn}(y) X_{m,ny}^{11} + a_{ij11}(y))_{,jy} = 0 \quad \text{in } P_1,$$

$$(a_{ijmn}(\mathbf{y}) X_{m,ny}^{11} + a_{ij11}(\mathbf{y})) n_j = 0 \quad \text{on } S,$$

$$\mathbf{X}^{11}(y) \text{ is periodic in } y_1 \text{ with period } T; \tag{3.8}$$

$$(a_{ijmn}(y)X^{2\alpha}_{m,ny}+a_{ij11}(y)y_\alpha)_{,jy}=0 \quad \text{in } P_1,$$
$$(a_{ijmn}(\mathbf{y})X^{2\alpha}_{m,ny}+a_{ij11}(\mathbf{y})y_\alpha)n_j=0 \quad \text{on } S, \tag{3.9}$$
$\mathbf{X}^{2\alpha}(y)$ is periodic in y_1 with period T;

$$(a_{ijmn}(\mathbf{y})X^{3}_{m,ny}+a_{ij\gamma 1}(\mathbf{y})s_\gamma y_\Gamma)_{,jy}=0 \quad \text{in } P_1,$$
$$(a_{ijmn}(\mathbf{y})X^{3}_{m,ny}+a_{ij\gamma 1}(\mathbf{y})s_\gamma y_\Gamma)n_j=0 \quad \text{on } S, \tag{3.10}$$
$\mathbf{X}^3(\mathbf{y})$ is periodic in y_1 with period T.

Here S means the lateral (free) surface of PC P_1, n is normal to S (see Fig. 1).

The solution of the problem can be expressed through the functions $\mathbf{X}^{11}(\mathbf{y})$, $X^{2\alpha}(\mathbf{y})$, $\mathbf{X}^3(\mathbf{y})$ as follows

$$\mathbf{u}^{(2)}=-\mathbf{X}^{2\alpha}(\mathbf{y})u^{(0)}_{\alpha,1x1x}(x_1)-y_\alpha V(x_1)\alpha_{,1x}(x)_1\mathbf{e}_1+\mathbf{X}^{11}(y)V_{1,1x}(x_1)+\mathbf{X}^3(y)\phi_{,1x}(x_1). \tag{3.11}$$

The proof of the formula (3.11) one can find in the work by Kolpakov (1991). One can verify the formula (3.11) without applying the mentioned paper, substituting eqn (3.11) into eqns (2.3) ($p=-3$), (3.5) ($k=1$).

Note. The first term in the right-hand side of eqn (3.6) (corresponding to bending) is a partial solution of the PC eqn (3.9). The second term in the right-hand side of eqn (3.6) (corresponding to torsion) is a solution of the uniform problem corresponding to the PC eqn (3.9), $U(y)$ is the solution of the uniform problem corresponding PC eqn (3.9). Then, from the mathematical point of view, the torsion is due to the degeneration of the PC eqn (3.9) in the local variables in the domain corresponding to the beam. No situations like this were mentioned early in the papers devoted to the homogenization analysis. The PCs, considered in connection with the monolithic composite and plate, are not degenerated (see e.g. Bensoussan *et al.*, 1978 for the monolithic composites, Caillerie, 1984—for plates).

Substituting eqn (3.11) into eqn (3.7) we have

$$\sigma^{(-3)}_{ij}=(a_{ijmn}(\mathbf{y})X^{11}_{m,ny}+a_{ij11}(\mathbf{y}))V_{1,1x}(x_1)+(a_{ijmn}(\mathbf{y})X^{2\alpha}_{m,ny}+a_{ij11}(\mathbf{y}))u^{(0)}_{\alpha,1x1x}(x_1)$$
$$+(a_{ijmn}(\mathbf{y})X^{3}_{m,ny}+a_{ij\gamma 1}(\mathbf{y})s_\gamma y_\Gamma)\phi_{,1x}(x_1). \tag{3.12}$$

Averaging eqns (3.12) with $ij=11$ over the PC P_1, we obtain

$$N^{(-3)}_{11}=AV_{1,1x}+A^1_\alpha u^{(0)}_{\alpha,1x1x}+b\phi_{,1x}. \tag{3.13}$$

Multiplying eqns (3.12) with $j=1$ by y_β and averaging over the PC P_1, we obtain the equations which can be broken down into two groups: equations for bending moments $M^{(-3)}_{1\beta}=\langle\sigma^{(-3)}_{11}y_\beta\rangle$ and torsion moments $M=M^{(-3)}_{32}-M^{(-3)}_{23}=\langle\sigma^{(-3)}_{31}y_2\rangle-\langle\sigma^{(-3)}_{21}y_3\rangle$:

$$M^{(-3)}_{1\beta}={}^1A_\beta V_{1,1x}+A^2_{\beta\alpha}u^{(0)}_{\alpha,1x1x}+B_\beta\phi_{,1x},\quad M=A^1V_{1,1x}+A^2_\alpha u^{(0)}_{\alpha,1x1x}+B\phi_{,1x}. \tag{3.14}$$

The coefficients in the eqns (3.13) and (3.14) are the following

$$A=\langle a_{1111}(\mathbf{y})+a_{11mn}(\mathbf{y})X^{11}_{m,ny}\rangle,$$
$$A^1_\alpha=\langle a_{1111}(\mathbf{y})y_\alpha+a_{11mn}(\mathbf{y})X^{2\alpha}_{m,ny}\rangle,$$
$$b=\langle a_{11\gamma\gamma}(\mathbf{y})s_\gamma y_\Gamma+a_{11mn}(\mathbf{y})X^{3}_{m,ny}\rangle,$$
$${}^1A_\beta=\langle(a_{1111}(\mathbf{y})+a_{11mn}(\mathbf{y})X^{11}_{m,ny})y_\beta\rangle,$$
$$A^2_{\alpha\beta}=\langle(a_{1111}(\mathbf{y})y_\alpha+a_{11mn}(\mathbf{y})X^{2\alpha}_{m,ny})y_\beta\rangle,$$

$$B_\beta = \langle (a_{11\gamma 1}(\mathbf{y}) s_\gamma y_\Gamma + a_{11mn}(\mathbf{y}) X^3_{m,ny}) y_\beta \rangle,$$

$$B = \langle a_{31\gamma 1}(\mathbf{y}) y_\Gamma y_2 \rangle s_\gamma - \langle a_{21\gamma 1}(\mathbf{y}) y_\Gamma y_3 \rangle s_\gamma + \langle a_{31mn}(\mathbf{y}) X^3_{m,ny} y_2 \rangle - \langle a_{21mn}(\mathbf{y}) X^3_{m,ny} y_3 \rangle,$$

$$A^1 = \langle (a_{3111}(\mathbf{y}) + a_{31mn}(y) X^{11}_{m,ny}) y_2 \rangle - \langle (a_{2111}(\mathbf{y}) + a_{21mn}(\mathbf{y}) X^{11}_{m,ny}) y_3 \rangle,$$

$$A^2_\alpha = \langle (a_{3111}(\mathbf{y}) y_\alpha + a_{31mn}(y) X^{2\alpha}_{m,ny}) y_2 \rangle - \langle (a_{2111}(\mathbf{y}) y_\alpha + a_{21mn}(\mathbf{y}) X^{2\alpha}_{m,ny}) y_3 \rangle.$$

The obtained eqns (3.13) and (3.14) are asymptotically exact governing equations of the beam considered as a 1-D structure. In the case under consideration these do not depend on the initial stresses. The coefficients in the right-hand parts of eqns (3.13) and (3.14) are the stiffnesses of the beam. As we see, these are expressed through solutions of the PCs eqns (3.8), (3.9) and (3.10). Note, that the derived governing equations coincide with the classical ones in form, but the stiffnesses are calculated in other ways.

Let us denote the right-hand parts of the governing eqns (2.13) and (2.14) by $N(V_{1,1x}, u^{(0)}_{m1x1x}, \phi_{,1x})$, $M_\beta(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})$, $M(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})$, respectively. These notations will be used in the next sections.

Investigating the equilibrium equations for the case of the absence of initial stresses (when, due to $\sigma^{(-2)}_{ij} = \sigma^{(-2)}_{ji}$, we have $N^{(-2)}_{ij} = N^{(-2)}_{ji}$), one can exclude the quantities $N^{(-2)}_{ij}$ from eqns (2.5) and (2.6) using this symmetry with respect to the indices i and j, only, without any additional information about $N^{(-2)}_{ij}$, as it was done in the paper by Caillerie (1984). In our case $N^{(-2)}_{ij}$ does not have the symmetry with respect to the indices i and j. Then we need to examine $N^{(-2)}_{ij}$ in detail, to obtain some additional information on them.

Let us insert $u^{(1)}$ into eqn (3.4) in accordance with eqn (3.6). Then we obtain

$$\sigma^{(-2)}_{ij} = a_{ijm1} u^{(2)}_{m,1x} + a_{ijmn} u^{(3)}_{m,ny} + b^{(-2)}_{ij\alpha 1} u^{(0)}_{\alpha,1x}(x_1) + b^{(-2)}_{ij\gamma\Gamma} s_\gamma \phi(x_1).$$

The first and second terms in the right-hand side of the present equation are symmetric with respect to i and j by virtue of symmetry of the elastic constants ($a_{ijmn} = a_{jimn}$). Then the relation below takes places

$$\sigma^{(-2)}_{ij} - \sigma^{(-2)}_{ji} = b^{(-2)}_{ij\alpha 1} u^{(0)}_{\alpha,1x}(x_1) - b^{(-2)}_{ji\alpha 1} u^{(0)}_{\alpha,1x}(x_1) + b^{(-2)}_{ij\gamma\Gamma} s_\gamma \phi(x_1) - b^{(-2)}_{ji\gamma\Gamma} s_\gamma \phi(x_1).$$

Averaging this equality over PC P_1, we obtain

$$N^{(-2)}_{ij} - N^{(-2)}_{ji} = \langle b^{(-2)}_{ij\alpha 1} \rangle u^{(0)}_{\alpha,1x} - \langle b^{(-2)}_{ji\alpha 1} \rangle u^{(0)}_{\alpha,1x} + \langle b^{(-2)}_{ij\gamma\Gamma} \rangle s_\gamma \phi - \langle b^{(-2)}_{ji\gamma\Gamma} \rangle s_\gamma \phi. \quad (3.15)$$

To continue our investigation, we need some facts about the average values of the initial stresses.

Proposition 1. Let the initial stresses σ^*_{ij}, which can be presented as $\sigma^*_{ij} = \varepsilon^{-3} \sigma^{*(-3)}_{ij} + \varepsilon^{-2} \sigma^{*(-2)}_{ij} + \cdots$, satisfy the equilibrium equations:

$$\partial \sigma^*_{ij} / \partial x_j = \varepsilon^a f_i \quad \text{in } Q_\varepsilon, \quad \sigma^*_{ij} n_j = \varepsilon^b g_i \quad \text{on } S_\varepsilon. \quad (3.16)$$

Then

(1) if $a \neq -4$, $b \neq -3$, then $\langle b^{(-3)}_{ijmn} Y_{,ny} \rangle = 0$;
(2) if $\sigma^{*(-3)}_{ij} = 0$ and $a \neq -3$, $b \neq -2$, then $\langle b^{(-2)}_{ijmn} Y_{,ny} \rangle = 0$

for every differentiable function $Y(y)$ periodic in y_1 with the period T.

Taking into account the definition of $b^{(-3)}_{ijmn}$, one must prove that $\langle \sigma^{*(-3)}_{jn} Y_{,ny} \rangle = 0$. Substituting the expression $\sigma^*_{ij} = \varepsilon^{-3} \sigma^{*(-3)}_{ij} + \varepsilon^{-2} \sigma^{*(-2)}_{ij} + \cdots$ for the initial stresses into eqn (3.16), with allowance for eqn (3.1), we obtain

$$\sigma^{*(-3)}_{ij,jy} = F_i \quad \text{in } P_1 \quad (F_i = 0 \quad \text{if } a \neq -4, F_i = f_i \quad \text{if } a = -4),$$

$$\overset{*}{\sigma}{}^{(-3)}_{ij} n_j = G_i \quad \text{on } S \quad (G_i = 0 \quad \text{if } a \neq -3, G_i = g_i \quad \text{if } a = -3),$$

$$\sigma^{*(-3)}_{ij}(x_1, \mathbf{y}) \text{ is periodic in } y_1 \text{ with period } T. \tag{3.17}$$

Let us consider the quantity $\langle \sigma^{(-3)}_{jn} Y_{,ny} \rangle$. Taking into account the definition of the average value over the PC P_1, and integrating by parts, one can find that this quantity is equal to

$$-\int_{P_1} \sigma^{*(-3)}_{jn,ny} Y \,\mathbf{dy} + \int_S \sigma^{*(-3)}_{jn} n_n Y \,\mathbf{dy} + \int_\omega \sigma^{*(-3)}_{j1} n_1 Y \,\mathbf{dy}.$$

Here ω means the opposite faces of PC P_1 normal to Oy_1-axis (Fig. 1). The integrals over P_1 and S are equal to zero as a consequence of eqn (3.17) (if $a \neq -4, b \neq -3$). The integral over ω is equal to zero as a consequence of the periodicity of the functions $\sigma^{*(-3)}_{jn}$ and Y and antiperiodicity of the vector-normal in y_1 (see Fig. 1). This proves statement (1) of the proposition. Analogously, one can prove statement (2) of the proposition.

Proposition 2. Under the conditions of proposition 1, $\langle \sigma^{*(-3)}_{i\alpha} \rangle = 0$ and $\langle \sigma^{*(-3)}_{1\alpha} y_\alpha \rangle = 0$, and $\langle \sigma^{*(-2)}_{i\alpha} \rangle = 0$ when $\sigma^{*(-3)}_{ij} = 0$.

Proposition 2 is a consequence of proposition 1. To prove the first and the third equations, one can put $Y = y_\alpha$, and $Y = y_\alpha^2$ to prove the second one.

Now, we can exclude the quantities $N^{(-2)}_{ij}$ from the equilibrium equations.

Bending. Consider the following equilibrium eqns (2.4) and (2.5). To exclude $N^{(-2)}_{1\alpha}$, apply the relations, which follow from eqn (3.15) and the definition of $b^{(-2)}_{ijmn}$ eqn (1.4):

$$N^{(-2)}_{1\beta} - N^{(-2)}_{\beta 1} = K_\beta, \tag{3.18}$$

where K_β is defined by the formula

$$\begin{aligned} K_\beta &= \langle b^{(-2)}_{\beta 1 \alpha 1} \rangle u^{(0)}_{\alpha,1x} - \langle b^{(-2)}_{1\beta\alpha 1} \rangle u^{(0)}_{\beta,1x} + \langle b^{(-2)}_{\beta 1 \gamma \Gamma} \rangle s_\gamma \phi - \langle b^{(-2)}_{1\beta\gamma\Gamma} \rangle s_\gamma \phi \\ &= \langle \sigma^{*(-2)}_{11} \rangle \delta_{\beta\alpha} u^{(0)}_{\alpha,1x} + \langle \sigma^{*(-2)}_{1\Gamma} \rangle \delta_{\beta\gamma} s_\gamma \phi. \end{aligned} \tag{3.19}$$

Differentiating eqn (2.5) and using eqns (2.4) and (3.18), we obtain

$$M_\beta(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})_{,1x1x} = K_{\beta,1x}. \tag{3.20}$$

Under conditions from proposition 1 ($a \neq -3, b \neq -2$) $K_\beta = \langle \sigma^{*(-2)}_{11} \rangle u^{(0)}_{\beta,1x}$, then the eqn (3.20) takes the classical form

$$M_\beta(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})_{,1x1x} = N^{*(-2)}_{11} u^{(0)}_{\beta,1x} \tag{3.21}$$

where $N^{*(-2)}_{11} = \langle \sigma^{*(-2)}_{11} \rangle$ is the initial axial force.

Torsion. Here, as above, we encounter a situation connected with the asymmetry of $N^{(-2)}_{ij}$, see eqn (2.6). In accordance with eqn (3.15) and the definition of $b^{(-2)}_{ijmn}$ eqn (1.4), we obtain

$$N^{(-2)}_{23} = N^{(-2)}_{32} + K,$$

where K is defined by the formula

$$K = N^{(-2)}_{32} - N^{(-2)}_{23} = \langle \sigma^{*(-2)}_{31} \delta_{2\alpha} + \sigma^{*(-2)}_{21} \delta_{3\alpha} \rangle u^{(0)}_{\alpha,1x} + \langle \sigma^{*(-2)}_{22} + \sigma^{*(-2)}_{33} \rangle \phi.$$

Then eqn (2.6) can be rewritten in the form

$$M(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})_{,1x} - K = 0. \tag{3.22}$$

Under conditions from proposition 1 ($a \neq -3, b \neq -2$) $K = 0$. It means, that the initial stresses of the order ε^{-2} do not influence the torsion of the beam.

4. THE CASE OF ZERO AXIAL FORCE, NON-ZERO MOMENTS

We consider now the case $\{\sigma^{*(-3)}_{ij} \neq 0, \sigma^{*(-2)}_{ij} = 0\}$ and $\sigma^{*(-3)}_{ij}$ satisfy the condition (1.6)—zero axial force.

Substituting eqns (2.1) and (2.2) into the local governing eqn (1.5), with allowance for eqn (3.1), we obtain in the case under consideration

$$\sum_{p=-4}^{\infty} \varepsilon^p \sigma^{(p)}_{ij} = \sum_{k=0}^{\infty} \varepsilon^k (\varepsilon^{-4} a_{ijmn} + \varepsilon^{-3} b^{(-3)}_{ijmn})(u^{(k)}_{m,1x}\delta_{n1} + \varepsilon^{-1} u^{(k)}_{m,ny}). \tag{4.1}$$

Equating the terms with identical power of ε in eqn (4.1), we obtain

$$\begin{aligned}
\sigma^{(-4)}_{ij} &= a_{ijm1}(\mathbf{y}) u^{(0)}_{m,1x} + a_{ijmn}(\mathbf{y}) u^{(1)}_{m,ny},\\
\sigma^{(-3)}_{ij} &= a_{ijm1}(\mathbf{y}) u^{(1)}_{m,1x} + b^{(-3)}_{ijm1}(\mathbf{y}) u^{(0)}_{m,1x} + a_{ijmn}(\mathbf{y}) u^{(2)}_{m,ny} + b^{(-3)}_{ijmn}(\mathbf{y}) u^{(1)}_{m,ny},\\
\sigma^{(-2)}_{ij} &= a_{ijm1}(\mathbf{y}) u^{(2)}_{m,1x} + b^{(-3)}_{ijm1}(\mathbf{y}) u^{(1)}_{m,1x} + a_{ijmn}(\mathbf{y}) u^{(3)}_{m,ny} + b^{(-3)}_{ijmn}(\mathbf{y}) u^{(2)}_{m,ny}.
\end{aligned} \tag{4.2}$$

Let us consider problem (2.7) ($p = -4$) with $\sigma^{(-4)}_{ij}$ given by eqn (4.2) and condition (3.5) ($k = 1$). As above, its solution $\mathbf{u}^{(1)}$ is given by the formula (3.6).

Substitution of eqn (3.6) into eqn (4.2) gives the following equations

$$\begin{aligned}
\sigma^{(-4)}_{ij} &= 0,\\
\sigma^{(-3)}_{ij} &= a_{ijmn}(\mathbf{y}) u^{(2)}_{m,ny} + a_{ij11}(\mathbf{y}) y_\alpha u^{(0)}_{\alpha,1x1x}(x_1) + a_{ij11}(\mathbf{y}) V_{1,1x}(x_1)\\
&\quad + a_{ij\gamma 1}(\mathbf{y}) s_\gamma y_\Gamma \phi(x_1)_{,1x} + (b^{(-3)}_{ij\alpha 1}(\mathbf{y}) - b^{(-3)}_{ij1\alpha}(\mathbf{y})) u^{(0)}_{\alpha,1x}(x_1) + b^{(-3)}_{ij\gamma\Gamma} s_\gamma \phi(x_1).
\end{aligned} \tag{4.3}$$

Under conditions of the propositions 1, see eqn (3.17), $b^{(-3)}_{ijmn,jy} = \sigma^{*(-3)}_{nj,jy}\delta_{im} = 0$ in P_1 and $b^{(-3)}_{ijmn} n_j = \sigma^{*(-3)}_{nj}\delta_{im} n_j = 0$ on S. Then, the terms containing $b^{(-3)}_{ijmn}$ have no influence on the solution, and, solving the problem (2.7) with $\sigma^{(-3)}_{ij}$ given by eqn (4.3), one can put

$$\sigma^{(-3)}_{ij} = a_{ijmn}(\mathbf{y}) u^{(2)}_{m,ny} + a_{ij11}(\mathbf{y}) y_\alpha u^{(0)}_{\alpha,1x1x}(x_1) + a_{ij11}(\mathbf{y}) V_{1,1x}(x_1) + a_{ij\gamma 1}(\mathbf{y}) s_\gamma y_\Gamma \phi(x_1)_{,1x}.$$

That expression coincides with eqn (3.7). Then the solution of the problem under consideration is given by the formula (3.11). Substituting eqn (3.11) into the second equation from eqn (4.3), we obtain

$$\begin{aligned}
\sigma^{(-3)}_{ij} &= (a_{ijmn}(\mathbf{y}) X^{11}_{m,ny} + a_{ij11}(\mathbf{y})) V_{1,1x}(x_1) + (a_{ijmn}(\mathbf{y}) X^{2\alpha}_{m,ny} + a_{ij11}(\mathbf{y})) u^{(0)}_{\alpha,1x1x}(x_1)\\
&\quad + a_{ijmn}(\mathbf{y}) X^3_{m,ny} + a_{ij\gamma 1}(\mathbf{y}) s_\gamma y_\Gamma)\phi_{,1x}(x_1) + (b^{(-3)}_{ij\alpha 1}(\mathbf{y}) - b^{(-3)}_{ij1\alpha}(\mathbf{y})) u^{(0)}_{\alpha,1x}(x_1) + b^{(-3)}_{ij\gamma\Gamma} s_\gamma \phi(x_1).
\end{aligned} \tag{4.4}$$

Averaging eqn (4.4) with $ij = 11$ over the PC P_1 and taking into account the definition of $b^{(-3)}_{ijkl}$ and propositions 2 and 3, we obtain

$$N^{(-3)}_{11} = N(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x}), \tag{4.5}$$

where $N(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})$ is determined in Section 3.

Multiplying eqn (4.4) with $j = 1$ by y_β and averaging over the PC P_1, we obtain, taking into account the definition of $b^{(-3)}_{ijkl}$ eqn (1.4), definition of the bending and torsional moments, condition (1.5) and propositions 2 and 3.

$$M_{1\beta}^{(-3)} = M_\beta(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x}) + C_{\beta\alpha}u^{(0)}_{\alpha,1x},$$
$$M = M(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x}) + C_\alpha u^{(0)}_{\alpha,1x} + J\phi. \tag{4.6}$$

Here $M_\beta(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})$, $M = M(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})$ are determined in Section 3. The coefficients on the right-hand side of eqn (4.6) are the following

$$C_{\beta\alpha} = \langle (b^{(-3)}_{11\alpha 1} - b^{(-3)}_{111\alpha})y_\beta\rangle = -\langle \sigma^{*(-3)}_{1\alpha} y_\beta\rangle,$$
$$C_\alpha = \langle (b^{(-3)}_{31\alpha 1} - b^{(-3)}_{311\alpha})y_2 - (b^{(-3)}_{21\alpha 1} - b^{(-3)}_{211\alpha})y_3\rangle$$
$$= \langle \sigma^{*(-3)}_{11} y_2\rangle\delta_{3\alpha} - \langle \sigma^{*(-3)}_{11} y_3\rangle\delta_{2\alpha},$$
$$J = \langle b^{(-3)}_{31\gamma\Gamma} s_\gamma y_2\rangle - \langle b^{(-3)}_{21\gamma\Gamma} s_\gamma y_3\rangle = -\langle \sigma^{*(-3)}_{12} y_2\rangle - \langle \sigma^{*(-3)}_{13} y_3\rangle. \tag{4.7}$$

Deriving eqn (4.6) and eqn (4.7), we take into account definition of $b^{(-3)}_{ijkl}$ eqn (1.4).

Let us exclude the quantities $N^{(-2)}_{ij}$ from the equilibrium eqns (2.4) and (2.5) (as it was noted, equilibrium equations are the same for all the cases under consideration). To do this, we examine $N^{(-2)}_{ij}$.

In the case under consideration [see eqn (4.2)] $\sigma^{(-2)}_{ij}$ is the sum of the terms symmetric with respect to i and j (the first and third terms) and the terms asymmetric with respect to i and j (the second and fourth ones). One can find that the relation below takes place

$$N^{(-2)}_{ij} = N^{(-2)}_{ji} + \langle (b^{(-3)}_{ijm1} - b^{(-3)}_{jim1})u^{(1)}_{m,1x}\rangle + \langle (b^{(-3)}_{ijmn} - b^{(-3)}_{jimn})u^{(2)}_{m,ny}\rangle. \tag{4.8}$$

Proposition 3. Under conditions of proposition 1 $\langle (b^{(-3)}_{ijmn} - b^{(-3)}_{jimn})u^{(2)}_{m,ny}\rangle = 0$.

To prove the proposition is enough put $Y = u^{(2)}_m$ into proposition 1.

Substituting eqn (3.6) into eqn (4.8) and taking into account the proposition 3, we can rewrite eqn (4.8) as

$$N^{(-2)}_{ij} = N^{(-2)}_{ji} + \langle (b^{(-3)}_{ij11} - b^{(-3)}_{ji11})y_\alpha\rangle u^{(0)}_{\alpha,1x1x} + \langle (b^{(-3)}_{ij\gamma 1} - b^{(-3)}_{ji\gamma 1})y_\Gamma\rangle s_\gamma \phi_{,1x}$$
$$+ \langle b^{(-3)}_{ijm1} - b^{(-3)}_{jim1}\rangle V_{m,1x}. \tag{4.9}$$

In accordance with proposition 1 and condition (1.5), the last term in the right-hand side of eqn (4.9) is equal to zero.

Applying the formula (4.9), one can exclude the quantities $N^{(-2)}_{ij}$ from the equilibrium equations.

Bending. Taking into account definition of $b^{(-3)}_{ijmn}$ eqn (1.4), one can derive from eqns (2.4) and (2.5) and eqns (4.6) and (4.9) the following equation

$$[M_\beta(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x}) + C_{\alpha\beta}u^{(0)}_{\alpha,1x}]_{,1x1x} = K_{\beta,1x}. \tag{4.10}$$

Here

$$K_\beta = N^{(-2)}_{1\beta} - N^{(-2)}_{\beta 1} = k_{\beta\alpha}u^{(0)}_{\alpha,1x1x} + k_\beta\phi_{,1x}, \tag{4.11}$$

where

$$k_{\beta\alpha} = \langle (b^{(-3)}_{\beta 111} - b^{(-3)}_{1\beta 11})y_\alpha\rangle = -\langle \sigma^{*(-3)}_{\beta 1} y_\alpha\rangle,$$
$$k_\beta = \langle (b^{(-3)}_{\beta 1\gamma 1} - b^{(-3)}_{1\beta\gamma 1})y_\Gamma\rangle s_\gamma = \langle \sigma^{*(-3)}_{11} y_\beta\rangle s_B \quad \text{(no sum in } \beta),$$

$B = 3$ if $\beta = 2$, and $B = 2$ if $\beta = 3$; $M_\beta(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})$ is determined in Section 3.

Torsion. One can derive from eqns (2.6), (4.6) and (4.9) the following equation

$$[M(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x}) + C_\alpha u^{(0)}_{\alpha,1x} + J\phi_{,1x}]_{,1x} = K.$$

Here

$$K = (N^{(-2)}_{32} - N^{(-2)}_{23}) = k\phi_{,1x}, \tag{4.12}$$

where

$$k = \langle (b^{(-3)}_{32\gamma 1} - b^{(-3)}_{23\gamma 1}) y_\Gamma s_\gamma \rangle = \langle \sigma^{*(-3)}_{21} y_2 + \sigma^{*(-3)}_{31} y_3 \rangle,$$

$M(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})$ is determined in Section 3.

Note, that all the coefficients determined by eqn (4.7), (4.11) and (4.12) can be written as the combinations of moments of the initial stresses $M^{*(-3)}_{i\alpha} = \langle \sigma^{*(-3)}_{i1} y_\alpha \rangle$:

$$C_{\beta\alpha} = M^{*(-3)}_{\alpha\beta}, \quad C_\alpha = M^{*(-3)}_{12}\delta_{3\alpha} - M^{*(-3)}_{13}\delta_{2\alpha}, \quad J = -M^{*(-3)}_{22} - M^{*(-3)}_{33},$$
$$k_{\beta\alpha} = -M^{*(-3)}_{\beta\alpha}, \quad k_\beta = M^{*(-3)}_{1\beta} s_B \quad (\text{no sum in } \beta), \quad k = M^{*(-3)}_{22} + M^{*(-3)}_{33}. \tag{4.13}$$

By virtue of symmetry of the initial stresses $\sigma^{*(-2)}_{ij}$ with respect to i and j, and proposition 2, $M^{*(-3)}_{\beta\beta} = 0$. Then $C_{\beta\beta} = 0$, $J = 0$, $k = 0$, and the non-zero coefficients in eqn (4.13) are the following

$$C_{\beta B} = -M^{*(-3)}_{B\beta}, \quad C_\alpha = M^{*(-3)}_{31}\delta_{2\alpha} - M^{*(-3)}_{21}\delta_{3\alpha},$$
$$k_{\beta B} = -M^{*(-3)}_{\beta B}, \quad k_\beta = M^{*(-3)}_{1\beta} s_B \quad (\text{no sum in } \beta, B). \tag{4.14}$$

The index B was determined in the previous section.

5. THE LIMIT 1-D PROBLEM

Let us write the obtained model for the special case, when the initial stresses are proportional parameters: $\sigma^{*(-2)}_{ij} = \lambda^{(-2)}\sigma^{0(-2)}_{ij}$, $\sigma^{*(-3)}_{ij} = \lambda^{(-3)}\sigma^{0(-3)}_{ij}$ ($\lambda^{(-2)}, \lambda^{(-3)}$ are the parameters) and satisfy the conditions of proposition 1. Then the equations of the limit model can be written in the following form:

In all the cases

$$N_{,1x} = 0. \tag{5.1}$$

In the case $\{\sigma^{*(-3)}_{ij} = 0, \sigma^{*(-2)}_{ij} \neq 0\}$:

$$M_\beta(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})_{,1x1x} = \lambda^{(-2)}[N^{*(-2)}_{11} u^{(0)}_{\alpha,1x}]_{,1x}. \tag{5.2}$$

In the case $\{\sigma^{*(-3)}_{ij} \neq 0, \sigma^{*(-2)}_{ij} = 0\}$ and eqn (1.6):

$$M_\beta(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})_{,1x1x} = \lambda^{(-3)}\{[-C_{\beta B} u^{(0)}_{B,1x}]_{,1x1x} + [k_{\beta B} u^{(0)}_{B,1x1x} + k_\beta \phi_{,1x}]_{,1x}\},$$
$$M(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})_{,1x} = \lambda^{(-3)}[C_\alpha u^{(0)}_{\alpha,1x}]_{,1x}. \tag{5.3}$$

$C_{\beta B}$, C_α, $k_{\beta B}$, k_β are determined by eqn (4.14); the functions N, M_β, M are determined in Section 3; no sum in β, B in eqn (5.3); the index B was determined above.

The boundary conditions are

$$V_1(a) = V_1(-a) = u^{(0)}_\alpha(a) = u^{(0)}_\alpha(-a) = u^{(0)}_{\alpha,1x}(a) = u^{(0)}_{\alpha,1x}(-a) = \phi(a) = \phi(-a) = 0$$

for all the cases.

Adding the eqns (5.2) and (5.3) one can obtain the two parameter ($\lambda^{(-2)}$ and $\lambda^{(-3)}$) model.

The problems (5.1)–(5.3) with the boundary conditions can be considered as an eigenvalue problem, describing the buckling of the beam. The models (5.1) and (5.2) coincide with the classical equations qualitatively.

Considering eqn (5.3) with the boundary equations, one finds, that in the case $\{\sigma^{*(-3)}_{ij} \neq 0, \sigma^{*(-2)}_{ij} = 0\}$—zero axial force—the buckling of the beam can be caused by the moments.

6. THE CASE OF THE UNIFORM HOMOGENIZED INITIAL STRESSES

Let the initial stresses depend on $\mathbf{x}/\varepsilon$, only (do not depend on x_1). Then, the average value over the PC P_1 does not depend on x_1 (see e.g. Kalamkarov and Kolpakov, 1997) and $C_{\beta\alpha}$, C_α, $k_{\beta\alpha}$, k_β, k are constants, and eqn (5.2) can be written as

$$M_\beta(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})_{,1x1x} = \lambda^{(-2)} N^{*(-2)}_{11} u^{(0)}_{\alpha,1x1x}, \tag{6.1}$$

and eqn (5.3) can be written as

$$M_\beta(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})_{,1x1x} = \lambda^{(-3)}[(-C_{\beta B}+k_{\beta B})u^{(0)}_{B,1x1x1x}+k_\beta \phi_{,1x1x}],$$
$$M(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})_{,1x} = \lambda^{(-3)} C_\alpha u^{(0)}_{\alpha,1x1x}. \tag{6.2}$$

Using eqn (4.14), one can find that $-C_{\beta B}+k_{\beta B} = M^{*(-3)}_{B\beta} - M^{*(-3)}_{\beta B} = M^{*(-3)} s_B$, and rewrite eqn (6.2) in the form

$$M_\beta(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})_{,1x1x} = \lambda^{(-3)}[-M^{*(-3)} s_\beta u^{(0)}_{B,1x1x1x} + M^{*(-3)}_{1\beta}\phi_{,1x1x}]s_B,$$
$$M(V_{1,1x}, u^{(0)}_{\alpha,1x1x}, \phi_{,1x})_{,1x} = \lambda^{(-3)}[M^{*(-3)}_{31} u^{(0)}_{2,1x1x} - M^{*(-3)}_{21} u^{(0)}_{3,1x1x}] \tag{6.3}$$

no sum in β, B here.

In eqn (6.3) $M^{*(-3)}_{B1}$ are bending moments, the $M^{*(-3)} = M^{*(-3)}_{32} - M^{*(-3)}_{23}$ is torsion moments corresponding to the initial stresses.

Note. The axial force $N^{*(-2)}_{11}$ and the moments $M^{*(-3)}_{1\beta}$, $M^{*(-3)}$ may be introduced both by averaging the local stresses $\sigma^{*(-2)}_{11}$ and the local moments $\sigma^{*(-3)}_{i1} y_\beta$ and by solving the 1-D problem describing the initial state of a beam with no initial stresses (i.e. the 1-D problem with no initial stresses). Both the methods give the same result. This follows from the paper by Kolpakov (1991). The first method incorporates a beam structure in an explicit form. The second method does not incorporate a beam structure explicitly. In the second method a beam structure is incorporated through the stiffnesses of the beam [see comment on the formulas (3.13) and (3.15)].

7. CYLINDRICAL BEAM: COMPARISON WITH THE CLASSICAL CASE

As it was demonstrated by Kolpakov (1994), for the classical cylindrical beam made of isotropic material the stiffnesses A, A^1_α, b, ${}^1A_\beta$, $A^2_{\alpha\beta}$, B_β, B coincide with ones derived on the basis of the plane section hypothesis. Then, eqns (5.1) and (5.2) coincide with the classical equations both qualitatively and quantitatively.

8. CONCLUSIONS

Application of the two-scale asymptotic method to the model of 3-D stressed body give a 1-D beam model.

The 1-D model has the form $Mx = Lx$, where M is a 1-D operator describing a beam with no initial stresses (the coefficients of the operator M, that is, stiffnesses are calculated on the basis solution of the cellular problems), and L is a 1-D operator incorporating the initial stresses.

The order of the initial stresses relative to the characteristic diameter of the beam plays a significant role in the problem. For the case of non-zero axial initial force the asymptotic analysis gives the classical 1-D beam model in which the operator L is expressed in terms of axial force corresponding to the initial stresses. For the case of zero axial initial force the asymptotic analysis gives a 1-D beam model in which the operator L is expressed in terms of moments of the initial stresses.

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